Université Jean Monnet – Saint-Étienne

Tutorial Sheet

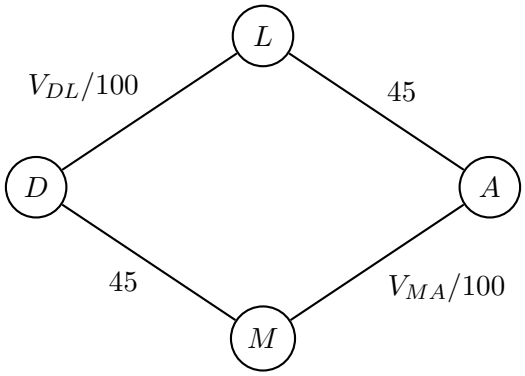
Game Theory

Microeconomics

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Exercise 1 – Braess's Paradox

4000 drivers located at point D want to reach destination A using the network below. Each driver seeks to minimize travel time.

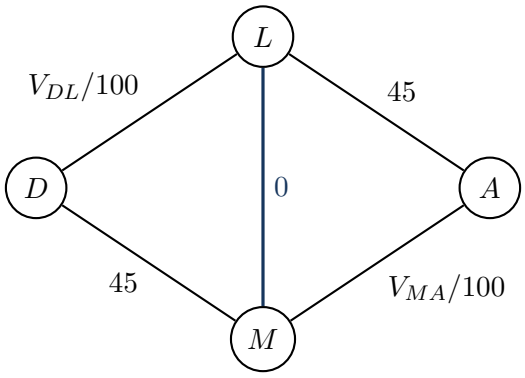


Road $D - L$ (respectively $M - A$) is subject to congestion. Its travel time, in minutes, equals the number of vehicles V_{DL} (respectively V_{MA}) using it, divided by 100. Roads $L - A$ and $D - M$ always take 45 minutes.

1. Model the situation as a strategic-form game. Specify the players, strategy sets, and payoffs.

2. Determine the pure-strategy Nash equilibria. What is the travel time at equilibrium?

A very fast road is now built between L and M, with travel time equal to zero.



3. Describe the new strategic-form game and determine its pure-strategy Nash equilibria.

4. Compare equilibrium travel times before and after the new road is built.

5. Explain the result. Why can adding a new option make every driver worse off?

Exercise 2 – Contribution to a Public Good

Providing a public good shared by a set N of agents requires individual contributions. Each agent $i \in N$ has m euros and chooses a contribution

$$x_i \in \{0, 1, \ldots, m\}.$$

The amount $m - x_i$ is invested in a private good with rate of return $r \in [0, 1[$. Agent i's payoff is

$$g_i(x_1, \ldots, x_n) = (1 + r)(m - x_i) + \sum_{j \in N} x_j.$$

1. Model the situation as a strategic-form game.
2. Interpret the two terms in the payoff function. What externality does one agent's contribution impose on the others?
3. Fix the contributions of all other agents. Compare agent i's payoff when contributing x_i with the payoff obtained by reducing the contribution by one unit.
4. Determine all pure-strategy Nash equilibria.
5. Compare the equilibrium outcome with the outcome in which every agent contributes m. Explain the difference.

Exercise 3 – Selling Pine Nuts

Selling pine nuts is more profitable when the nuts are harvested later. Harvesting takes place on land that is freely accessible to everyone. Suppose that $k \in \mathbb{N}^*$ harvesters compete. Each chooses a harvesting date in $\{1, 2, 3\}$.

Let $P(t)$ denote the total value obtained if all pine nuts are harvested at date t, with

$$0 < P(1) < P(2) < P(3).$$

Let t_0 be the earliest date chosen by at least one harvester. At date t_0, all pine nuts are harvested. The harvesters who chose t_0 share $P(t_0)$ equally; all other harvesters receive 0.

1. Represent this situation as a strategic-form game.
2. Find the Nash equilibria in each of the following cases:
   (a) $P(1) = 1$, $P(2) = 2$, $P(3) = 9$ and $k = 3$;
   (b) $P(1) = 1$, $P(2) = 2$, $P(3) = 9$ and $k = 10$;
   (c) $P(1) = 1$, $P(2) = 6$, $P(3) = 9$ and $k = 3$.
3. Characterize the Nash equilibria as a function of k and $P(1), P(2), P(3)$.
4. Explain the tension between waiting to increase the value of the resource and harvesting before the other players.

Exercise 4 – At the Show: Stand Up or Stay Seated

A theatre contains n spectators. Each spectator chooses either to remain seated ($x_i = 0$) or to stand up ($x_i = 1$), at a cost $\delta > 0$. Player i's payoff is

$$u_i(x_i, x_{-i}) = \begin{cases} 1 - \dfrac{\sum_{j \neq i} x_j}{n}, & \text{if } x_i = 0, \\ 2 - \delta - \dfrac{\sum_{j \neq i} x_j}{n}, & \text{if } x_i = 1. \end{cases}$$

1. Model the problem as a strategic-form game.
2. Write the definition of a pure-strategy Nash equilibrium for this game.
3. Compute every player's payoff when everyone remains seated.
4. Suppose $\delta < 1$. Determine the Nash equilibria and explain the result.
5. Suppose $\delta > 1$. Determine the Nash equilibria and explain the result.
6. What happens in the limiting case $\delta = 1$?

Exercise 5 – Diffusion of Responsibility

In an emergency, a person needs immediate assistance. A group of n individuals witnesses the situation. Each individual chooses whether to help ($x_i = 1$) or not ($x_i = 0$). Helping costs $c > 0$. If at least one person helps, every member of the group receives a benefit B/n.

Player i's payoff is

$$u_i(x_i, x_{-i}) = \frac{B}{n}\left(1 - \prod_{j=1}^{n}(1 - x_j)\right) - cx_i.$$

1. Explain the payoff function and model the problem as a strategic-form game.
2. Determine the pure-strategy Nash equilibria when $c > B/n$.
3. Determine the pure-strategy Nash equilibria when $c < B/n$.
4. Study separately the case $c = B/n$.
5. Interpret the results in terms of free riding and diffusion of responsibility.
6. Propose a simple modification of the payoffs that could provide stronger incentives to help, and discuss its effect on equilibrium behavior.

Exercise 6 – A Tragedy of the Commons: When to Fish?

Two fishers, 1 and 2, each choose a fishing date $t_i \in [0, +\infty)$. As long as no fishing occurs, the fish stock evolves according to

$$X(t) = e^t, \qquad X(0) = 1.$$

Each fisher has harvesting capacity $q > 1$. If i fishes strictly before j, player i catches $\min\{q, X(t_i)\}$; the remaining stock then continues to grow exponentially until player j fishes. If both fish simultaneously at date t, each receives $X(t)/2$ when $X(t) < 2q$, and q when $X(t) \geq 2q$.

The discounted value of one fish harvested at date t is e^{-rt}, with $r \in]0,1[$. Player i's payoff is therefore

$$u_i(t_1, t_2) = e^{-rt_i} h_i(t_1, t_2),$$

where h_i is the quantity actually harvested. Let $t_q = \ln q$.

1. Write, without solving the game, the definition of a Nash equilibrium in this setting.
2. Show that if one player fishes before the other at a date $t < t_q$, that player catches the entire available stock.
3. Show that at equilibrium at least one player must fish no later than t_q.
4. Suppose $t_1 < t_2$ and $t_1 \leq t_q$. Show that the stock is zero immediately after player 1 fishes and that player 2 catches nothing.
5. Deduce that player 2 can improve their payoff by fishing at date t_1. Show that every pure-strategy Nash equilibrium must therefore be symmetric, of the form (t, t).
6. For $0 < t \leq t_q$, compute each player's payoff at (t, t) and show that a player can improve their payoff by fishing slightly earlier.
7. Verify that $(0, 0)$ is a Nash equilibrium and determine the set of pure-strategy Nash equilibria.
8. Explain the result and compare it with the pine-nut harvesting game.

Exercise 7 – The Median Voter Theorem

Consider an election with two candidates, denoted A and B. Each candidate $i \in \{A, B\}$ chooses an ideological position from

$$S_i = \{1, \ldots, 10\}.$$

Position 1 represents the leftmost position and position 10 the rightmost position on the ideological spectrum. Voters are distributed across the ten positions as follows:

Position	1	2	3	4	5	6	7	8	9	10
Voters (%)	5	30	14	2	4	5	5	25	5	5

Each voter votes for the candidate whose position is closest to their own. If the two candidates are equally distant from a given position, the voters at that position split their votes equally between the two candidates. Each candidate seeks to maximize their percentage of the vote. This percentage defines candidate i's payoff u_i. For example,

$$u_A(8, 8) = 50, \qquad u_A(7, 8) = 65.$$

This defines a two-player strategic-form game with players $\{A, B\}$, strategy sets $S_A = S_B = \{1, \ldots, 10\}$, and payoff functions u_A and u_B.

1. Show that, for each $i \in \{A, B\}$, strategy $1 \in S_i$ is strictly dominated by strategy $2 \in S_i$.
2. Are there any other strategies that strictly dominate strategy 1?
3. Using iterated elimination of strictly dominated strategies, show that this game has a unique Nash equilibrium.

4. Now modify the initial setting by introducing a third candidate. Suppose there are only five ideological positions, $\{1, 2, 3, 4, 5\}$, with voters uniformly distributed across them (20% at each position).

   (a) Model this situation as a strategic-form game.

   (b) Determine the set of Nash equilibria of this game.

Exercise 8 – Computing a Nash Equilibrium

Determine the Nash equilibria in pure and mixed strategies of the following strategic-form game:

	L	R
T	$(1, 1)$	$(0, 2)$
B	$(2, 0)$	$(-1, -1)$

Exercise 9 – Modified Matching Pennies

Determine the Nash equilibria in pure and mixed strategies of the following strategic-form game:

	H	T
H	$(1, -2)$	$(0, 3)$
T	$(-10, 7)$	$(5, -1)$

For each candidate equilibrium, ask: who can deviate, to which strategy, and why would the deviation improve that player's payoff?