Graph Theory and Applications

Session 5 — Two-player games and Grundy functions

Master 1 APE

1 Problem 1 — Two-player game

Recall that on a directed graph $G = (S, A)$, the neighborhood (or set of successors) of a vertex $x \in S$ is the set of vertices $y \in S$ such that $(x, y) \in A$. An element of this set is called a neighbor of x.

1.1. In this question, we present a variant of the Nim game, known in China as fan-tan and in Africa as tiouk-tiouk. Imagine a bag containing n coins, $n \geq 1$. Two players a (the player who plays first) and b take turns removing a certain number of coins, at least one and at most two. The player who cannot take a coin loses (and thus the other player wins). **Note that the objective of the game is not to obtain the maximum number of coins but to take the last one**. For each of the following cases, briefly explain whether it is better to be the first or the second player to move.

(a) $n = 1$

(b) $n = 2$

(c) $n = 3$

1.2. This type of game (a two-player turn-based game) can be represented by a directed graph $G = (S, A)$ as follows. At the beginning, the position of the game is represented by the vertex x_0. Player a must choose a possible position x_1 which belongs to one of the neighbors of x_0, giving the new position of the game. Then, player b must choose a position x_2 which belongs to one of the neighbors of x_1, giving the new position of the game, and so on, until one of the players cannot move because the previous player has placed the game in a position x with no neighbors. This game is called a **direct game** if the player who can no longer play (being on a vertex with no neighborhood) loses.

(a) In the example in question 1, S represents the numbers $\{0, \ldots, n\}$, x_0 is the vertex n, representing the number of coins in the bag at the start of the game. Since the first player can take one or two coins, the neighbors of x_0 are the vertices $n-1$ and $n-2$ (if $n \geq 2$). More generally, the arcs in this game are represented by $A = \{(i, j) \in S^2 \text{ such that } i - j \in \{1, 2\}\}$. Represent this graph when $n = 3$ (note, there are 4 vertices!).

(b) Is this game a direct game?

1.3. We say that a directed graph $G = (S, A)$ is **progressively finite** if there is no infinite walk in G. Thus, any game on a progressively finite graph stops after a certain number of steps – and therefore has a winner and a loser. If the graph is not progressively finite, there exist games that never end.

(a) Suppose that S is finite, and that there exists an infinite walk $(x_0, x_1, \ldots)$ in G. Show that G contains a cycle.

(b) Deduce that any graph without a cycle and having a finite number of vertices is progressively finite.

1.4. Let $G = (S, A)$ be a graph with possibly an infinite number of vertices. In this question, we assume that there exists a function $\phi : S \to \mathbb{N}$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$ such that for every $x \in S$ and every y in the neighborhood of x, $\phi(x) > \phi(y)$.

(a) Let $(x_0, \ldots, x_n)$ be a walk in G of length n. Show (this is an easy question) that

$$\phi(x_0) \geq \phi(x_0) - \phi(x_n).$$

(b) By noting, without proving it, that

$$\phi(x_0) - \phi(x_n) = (\phi(x_0) - \phi(x_1)) + (\phi(x_1) - \phi(x_2)) + \ldots + (\phi(x_{n-1}) - \phi(x_n)),$$

show that $\phi(x_0) \geq n$.

(c) Deduce that the existence of such a function ϕ implies that the graph is progressively finite.

(d) Does such a function exist for the graph constructed in question 2(a)?

1.5. Let $G = (S, A)$ be a simple directed graph. We say that a set $N \subseteq S$ is a **kernel** of G if it satisfies the following two properties:

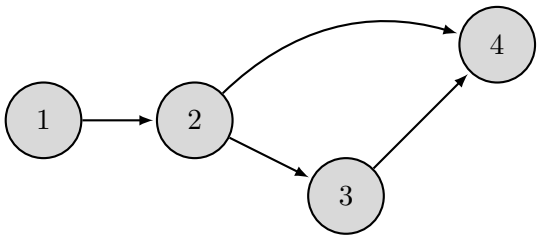
- For each x in N, none of the neighbors of x are in N.
- For each y outside of N, at least one of the neighbors of y is in N.

Show that in a game on a graph with a kernel N, if a player has the possibility of choosing a position $x \in N$ (re-read question 2 to understand what this means), this player has a strategy that allows them not to lose the game. **Note that not losing does not mean winning.**

1.6. Let $G = (S, A)$ be a directed graph. For every $x \in S$, let $V(x)$ be the neighborhood of x. We say that a function $g : S \to \mathbb{N}$ is a **Grundy function** for the graph G if for each x in S,

$$g(x) = \min\{n \in \mathbb{N},\, n \notin g(V(x))\}.$$

In other words, g associates to each vertex the smallest nonnegative integer that is different from the numbers associated with each of its neighbors, and 0 to vertices with no neighbors. For example, on the following graph,



$g(4) = 0$ because it has no neighbors, $g(3) = 1$ because 1 is the smallest integer not belonging to $\{0\} = g(\{4\})$, $g(2) = 2$ because 2 is the smallest integer not belonging to $\{0, 1\} = g(\{3, 4\})$, and $g(1) = 0$ because 0 is the smallest integer not belonging to $\{2\} = g(\{2\})$.

(a) Find the Grundy function for the graph constructed in question 2(a).

(b) Show that if g is a Grundy function for G, then

$$N = \{x \in S,\, g(x) = 0\}$$

is a kernel of G.

(c) Explain how to win the game presented in question 1.

(d) On a finite directed graph without cycles, show that a Grundy function exists and is unique: start with vertices that have no successors, then proceed backwards. *You may complete this question at home.*