Graph Theory and Applications

Session 6 — Can we draw a network without crossings?

Master 1 APE

1 Problem 1 — Can we draw a network without crossings?

Let G be a finite simple undirected graph with $s \geq 1$ vertices and a edges. A **tree** is a connected graph with no cycle. A **leaf** is a vertex of degree one.

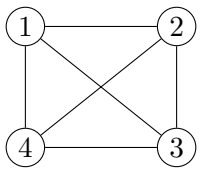
Question 1. Show that if G is a tree, then $a = s - 1$. You may use the following steps.

(a) Suppose $s \geq 2$, and choose a path with as many edges as possible. Show that each endpoint is a leaf. Consider what would happen if an endpoint had another neighbour on the path, or outside it.

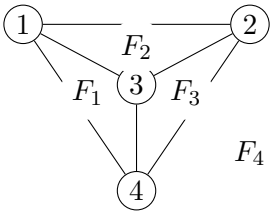
(b) Delete a leaf and its incident edge. Show that the remaining graph is still a tree.

(c) Conclude by induction on s, beginning with $s = 1$.

A graph is **planar** if it admits a drawing in the plane in which edges meet only at common endpoints. Such a drawing is called a **plane representation**. For instance, the same graph is shown below first with crossings, then without them.



A drawing with crossings



A plane representation

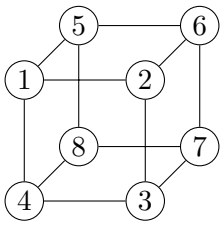
Vertices and edges may change their positions in the plane. A crossing in a drawing is not an additional vertex.

Fix a plane representation. Its **faces** are the connected regions of the plane left after removing the drawing, including the unbounded region. Let f be their number. The right-hand drawing has three bounded faces and one unbounded face.

Question 2. Verify that, for this graph and its plane representation,

$$s - a + f = 2.$$

Question 3. Show that the graph below is planar and prove that, for the plane representation you provide, one again has $s - a + f = 2$.



Question 4. Show that any plane representation of a tree has exactly one face, and deduce that in this situation $s - a + f = 2$. You may use the following elementary fact without proof: adding a leaf and its incident edge to a plane drawing creates no new face. Argue by induction, using Question 1.

Proving the formula

We now argue **by contradiction** and assume:

> There exists a connected graph together with a plane representation such that $s-a+f \neq 2$.

Let $a^\star$ be the **smallest possible number of edges** of such a counterexample. Let $G^\star$ have $s^\star$ vertices, $a^\star$ edges and a plane representation with $f^\star$ faces such that $s^\star - a^\star + f^\star \neq 2$.

Question 5. Show that $G^\star$ is not a tree and deduce the existence of a cycle. Denote by $e^\star$ an edge of this cycle.

Question 6. We admit, without proof, the following elementary property of plane drawings: an edge belonging to a cycle has a different face on each of its two sides; deleting that edge merges these faces and decreases the number of faces by one. Illustrate this property using a graph of your choice.

Question 7. Remove $e^\star$ from $G^\star$, retaining all vertices. Let $\widetilde{G}$ be the resulting graph, and let $\widetilde{s}, \widetilde{a}$ denote its numbers of vertices and edges.

(a) What are $\widetilde{s}$ and $\widetilde{a}$?

(b) Show that $\widetilde{G}$ is connected. Use the remaining edges of the cycle to replace any use of $e^\star$.

(c) Show that the inherited plane representation of $\widetilde{G}$ has $\widetilde{f} = f^\star - 1$ faces.

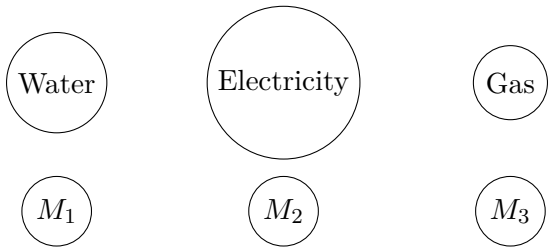
(d) Deduce that
$$\widetilde{s} - \widetilde{a} + \widetilde{f} = s^\star - a^\star + f^\star.$$

(e) Use the minimality of $a^\star$ to obtain a contradiction.

We have therefore proved **Euler's formula**: every plane representation of a finite connected planar graph satisfies
$$\boxed{s - a + f = 2.}$$

Can we connect three houses to three suppliers?

Question 8. We wish to connect three houses M_1, M_2, M_3 to water, electricity and gas. Neither the houses nor the suppliers are connected to each other. Complete this graph so that it represents the problem.



To decide whether the cables can be drawn without crossings, **suppose that this graph has a plane representation**. In Questions 9–11, f refers to that hypothetical representation.

The **length of a face boundary** counts edge occurrences along its boundary. An edge is counted once on each side, even when both sides border the same face.

Question 9. Show that every cycle of this graph has at least four edges. Show also that each edge belongs to a cycle of length four. Using the property in Question 6, deduce that every face boundary has length at least four. *Hint: the vertices on a cycle alternate between houses and suppliers. A boundary of length two would traverse the same edge in both directions.*

Question 10. By counting edge sides along all face boundaries, deduce

$$4f \leq 2a, \qquad \text{hence} \qquad 2f \leq a.$$

Question 11. Deduce that it is impossible to solve the problem in Question 8 without crossing cables. Use Euler's formula and the values of s and a for this graph.