Game Theory

Worked Solutions: Exercises 5–9

Microeconomics · Undergraduate (L3)
Université Jean Monnet – Saint-Étienne

Academic year 2026–2027

These notes follow the tutorial sheet. Every equilibrium claim is justified by a comparison with unilateral deviations. In Exercises 8 and 9, graphical best-response analysis complements the algebra.

Exercise 5 — Diffusion of Responsibility

There are n bystanders. Each chooses $x_i \in \{0, 1\}$, where 1 means helping. The payoff is

$$u_i(x) = \frac{B}{n}\left(1 - \prod_{j=1}^{n}(1 - x_j)\right) - cx_i, \qquad B > 0,\ c > 0.$$

1. The product equals 1 exactly when nobody helps. Thus the benefit B/n is obtained by *every* player as soon as at least one player helps; only helpers incur c. The normal-form game is $N = \{1, \ldots, n\}$, $S_i = \{0, 1\}$, with payoffs u_i as above.

2. If $c > B/n$, helping when nobody else helps yields $B/n - c < 0$, whereas not helping yields 0. If someone else helps, helping yields $B/n - c < B/n$, the payoff from not helping. Thus $x_i = 0$ is strictly dominant and the *unique* Nash equilibrium is $(0, \ldots, 0)$.

3. If $c < B/n$, the profile with nobody helping is not an equilibrium: any player gains $B/n - c > 0$ by helping. A profile with exactly one helper *is* an equilibrium: that helper earns $B/n - c > 0$ instead of 0, while each nonhelper earns B/n and would lose c by joining. A profile with two or more helpers is not an equilibrium: any helper can stop helping and save c without eliminating the benefit. Hence there are exactly n pure equilibria, each with one helper.

4. If $c = B/n$, the no-help profile *is* an equilibrium: a unilateral helper would earn 0 as well. Every profile with exactly one helper is also an equilibrium: the helper is indifferent between helping and not helping; nonhelpers strictly prefer not to help. Profiles with at least two helpers are not equilibria. There are therefore $n + 1$ pure equilibria.

5. The benefit is non-excludable within the group, but the cost is private. Once someone helps, everyone else strictly prefers to free ride. When $c < B/n$, someone helps in every pure equilibrium, but no one wants to be a second helper. When $c > B/n$, even being the sole helper is unattractive.

6. For example, reward a helper with an additional private benefit $s \geq 0$: $\widetilde{u}_i = u_i + sx_i$. If $s > c$, helping is strictly dominant, since the net private reward $s - c$ is positive even when someone else already helps; the unique equilibrium is universal helping. If $0 < s < c$, helping remains costly when others help, but the condition for a sole-helper equilibrium becomes $c - s \leq B/n$. This illustrates how a subsidy can change incentives without necessarily making universal helping efficient.

Exercise 6 — A Tragedy of the Commons: When to Fish?

The stock is $X(t) = e^t$, each fisher has capacity $q > 1$, and $0 < r < 1$. Let $t_q = \ln q$ and $u_i = e^{-rt_i} h_i$.

1. A pair $(t_1^*, t_2^*) \in [0, \infty)^2$ is a Nash equilibrium if, for both $i = 1, 2$ and all $t_i \geq 0$, $u_i(t_i^*, t_j^*) \geq u_i(t_i, t_j^*)$.

2. If fisher i arrives first at $t < t_q$, then $e^t < e^{t_q} = q$. Their capacity is enough to take the entire stock: $h_i = \min\{q, e^t\} = e^t$.

3. Suppose both dates exceed t_q. The earlier fisher, at $t_i > t_q$, catches q and obtains qe^{-rt_i}. By switching to t_q, that fisher catches q (since $X(t_q) = q$) and obtains $qe^{-rt_q} > qe^{-rt_i}$. If both choose the same $t > t_q$, a fisher can move slightly earlier to a date $s \in (t_q, t)$ and harvest q alone, earning qe^{-rs}. At the original tie, their catch is at most q, hence their payoff at t is at most $qe^{-rt} < qe^{-rs}$. Thus at least one player must fish by t_q.

4. If $t_1 < t_2$ and $t_1 \leq t_q$, the first fisher catches e^{t_1} and leaves zero fish. Since zero remains zero under exponential growth, $h_2 = 0$.

5. The late fisher can switch to t_1, sharing e^{t_1} and earning $e^{(1-r)t_1}/2 > 0$ instead of zero. Therefore no asymmetric profile with an earlier date at most t_q is an equilibrium. By part 3, *all* pure equilibria must be symmetric.

6. At (t, t) for $0 < t \leq t_q$, the total stock is $e^t \leq q < 2q$, so each receives $e^t/2$ and earns $\frac{1}{2}e^{(1-r)t}$. A deviation to $s < t$ harvests e^s alone and yields $e^{(1-r)s}$. As $s \uparrow t$, this tends to $e^{(1-r)t}$, strictly above the tie payoff. Hence a sufficiently small advance is profitable.

7. At $(0, 0)$, each catches $X(0)/2 = 1/2$ and earns $1/2$. If one fisher deviates to any $s > 0$, the other has already harvested $1/2$ at time zero. The remaining stock is $\frac{1}{2}e^s$, so the deviator catches $\min\{q, \frac{1}{2}e^s\}$ and earns

$$u_i(s, 0) = e^{-rs} \min\{q, \tfrac{1}{2}e^s\}.$$

For sufficiently small $s > 0$, this equals $\frac{1}{2}e^{(1-r)s} > \frac{1}{2}$ because $r < 1$. *Consequently* $(0, 0)$ *is not a Nash equilibrium under the stated stock-regrowth rule.* This contradicts the instruction in Question 7. In fact the game has *no pure-strategy Nash equilibrium*: parts 3–6 exclude all positive symmetric dates, while this deviation excludes $(0, 0)$.

8. The strategic pressure to preempt resembles the pine-nut game, but an important difference is that the fish stock *regrows after partial harvesting.* The sheet's requested conclusion $(0, 0)$ would hold if the second fisher received no fish after a simultaneous harvest at time zero (or if post-harvest regrowth were excluded), but not with the given dynamics.

Exercise 7 — The Median Voter Theorem

Let $w = (5, 30, 14, 2, 4, 5, 5, 25, 5, 5)$ be the percentages at positions $1, \ldots, 10$. A voter at v supports the nearer candidate, splitting equally in a tie. The two payoffs sum to 100.

1. Strategy 2 strictly dominates strategy 1. For every opposing position $b \geq 2$, moving from 1 to 2 moves closer to every voter at $v \geq 2$ and never loses a voter to the left of position 1; at least some positive mass switches or ceases to split. If $b = 1$, $u_A(1, 1) = 50$ while $u_A(2, 1) = 95$. Thus $u_A(2, b) > u_A(1, b)$ for every b; symmetry applies to candidate B.

2. Yes: positions $2, 3, 4, 5, 6$ each strictly dominate 1 in the original game. One can verify this directly by comparing the votes gained at each possible opposing position. Positions

$7, 8, 9, 10$ do not strictly dominate 1 (for example, against an opponent at 2, position 7 wins fewer votes than position 1).

3. For two candidates, the unique pure Nash equilibrium is $(4, 4)$. At $(4, 4)$ both earn 50. A move left to $a \leq 3$ against 4 wins at most the voters at positions 1–3, a total of 49% (and possibly a split at the midpoint); a move right to $a \geq 5$ wins at most the voters at positions 5–10, also 49%. Thus neither can profitably deviate. For a transparent iterative-elimination argument, first remove 1 and 10 (strictly dominated by 2 and 9 respectively); then successively remove extreme positions $2, 9$, followed by $3, 8$, then $7, 6, 5$, retaining only 4 for both players. These steps can be checked from the vote-share formula below; the appendix-style table gives an independent exhaustive verification of best responses.

4. With three candidates, let $S_i = \{1, 2, 3, 4, 5\}$ and give each location mass 20. At profile $s = (s_1, s_2, s_3)$, each voter splits its vote equally among all candidates at the minimum distance from that voter. Formally,

$$u_i(s) = 20 \sum_{v=1}^{5} \frac{\mathbf{1}\{i \in M(v, s)\}}{|M(v, s)|}, \quad M(v, s) = \arg \min_{j \in \{1,2,3\}} |v - s_j|.$$

There is *no pure Nash equilibrium.* This can be verified by checking the $5^3 = 125$ profiles. By symmetry, it suffices to check the 35 unordered triples $s_1 \leq s_2 \leq s_3$; each has at least one profitable unilateral deviation. In particular, all three at the center gives each $100/3$, but a candidate moving to position 2 obtains 40, and a candidate moving to 4 also obtains 40. More generally, adding a third candidate destroys the two-candidate convergence conclusion for this discrete uniform example.

Checking two-candidate payoffs. For any a, b, a useful formula is

$$u_A(a, b) = \sum_{v=1}^{10} w_v \left(\mathbf{1}\{|v - a| < |v - b|\} + \tfrac{1}{2}\mathbf{1}\{|v - a| = |v - b|\} \right).$$

Exercise 8 — Computing a Nash Equilibrium

	L	R
T	$(1, 1)$	$(0, 2)$
B	$(2, 0)$	$(-1, -1)$

Pure equilibria. Against L, player 1 prefers B ($2 > 1$); against R, player 1 prefers T ($0 > -1$). Against T, player 2 prefers R ($2 > 1$); against B, player 2 prefers L ($0 > -1$). Thus (B, L) and (T, R) are the two pure equilibria.

Mixed strategies. Let $p = \Pr(T)$ and $q = \Pr(L)$. Player 1's expected payoffs from pure actions are

$$U_1(T; q) = q, \qquad U_1(B; q) = 2q - (1 - q) = 3q - 1.$$

Player 1 is indifferent when $q = 1/2$. Likewise, player 2 obtains

$$U_2(L; p) = p, \qquad U_2(R; p) = 2p - (1 - p) = 3p - 1,$$

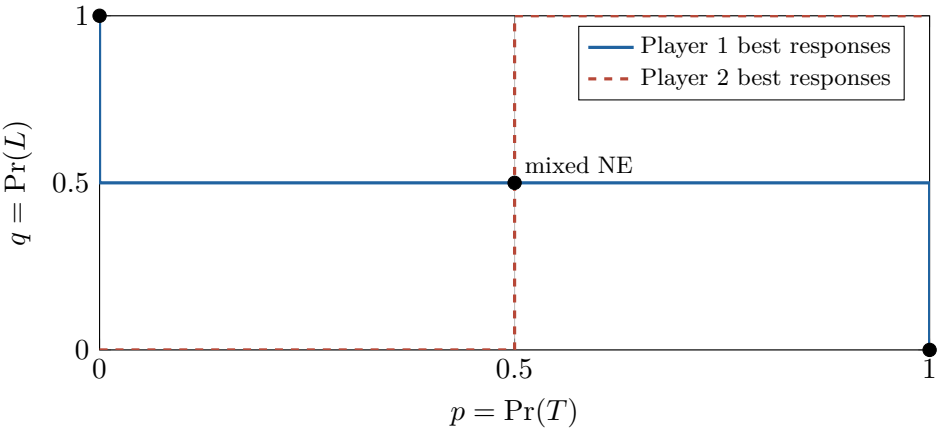
and is indifferent at $p = 1/2$. Hence the unique completely mixed equilibrium is

$$\boxed{(p^*, q^*) = (\tfrac{1}{2}, \tfrac{1}{2}).}$$

Best-response correspondences.

$$\mathrm{BR}_1(q) = \begin{cases} \{1\} & q < 1/2, \\ lbrack 0, 1] & q = 1/2, \\ \{0\} & q > 1/2, \end{cases} \qquad \mathrm{BR}_2(p) = \begin{cases} \{0\} & p < 1/2, \\ lbrack 0, 1] & p = 1/2, \\ \{1\} & p > 1/2. \end{cases}$$

Here $\mathrm{BR}_1(q)$ denotes the set of probabilities p of playing T; $\mathrm{BR}_2(p)$ denotes the set of probabilities q of playing L. The intersections are $(p, q) = (0, 1), (1, 0), (1/2, 1/2)$.



Reading the diagram: blue segments satisfy player 1's best-response condition, red dashed segments satisfy player 2's; their intersections are precisely the Nash equilibria.

Exercise 9 — Modified Matching Pennies

	H	T
H	$(1, -2)$	$(0, 3)$
T	$(-10, 7)$	$(5, -1)$

Pure strategies. If player 2 plays H, player 1 prefers H ($1 > -10$); if player 2 plays T, player 1 prefers T ($5 > 0$). If player 1 plays H, player 2 prefers T ($3 > -2$); if player 1 plays T, player 2 prefers H ($7 > -1$). Every cell permits a profitable deviation, so there is *no pure equilibrium.*

Mixed strategies. Let $p = \Pr_1(H)$ and $q = \Pr_2(H)$. For player 1,

$$U_1(H; q) = q, \qquad U_1(T; q) = -10q + 5(1 - q) = 5 - 15q.$$

Indifference requires $q = 5/16$. For player 2,

$$U_2(H; p) = -2p + 7(1 - p) = 7 - 9p, \qquad U_2(T; p) = 3p - (1 - p) = 4p - 1.$$

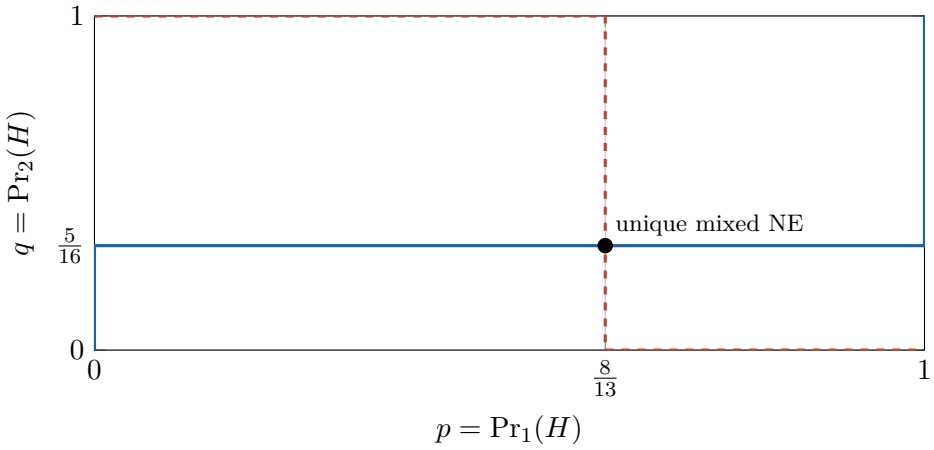
Indifference requires $p = 8/13$. Therefore the unique Nash equilibrium is

$$\boxed{(p^*, q^*) = (\tfrac{8}{13}, \tfrac{5}{16}).}$$

The equilibrium payoffs are $U_1^* = 5/16$ and $U_2^* = 7 - 9(8/13) = 19/13$.

Best-response correspondences.

$$\mathrm{BR}_1(q) = \begin{cases} \{0\} & q < 5/16, \\ lbrack 0, 1] & q = 5/16, \\ \{1\} & q > 5/16, \end{cases} \qquad \mathrm{BR}_2(p) = \begin{cases} \{1\} & p < 8/13, \\ lbrack 0, 1] & p = 8/13, \\ \{0\} & p > 8/13. \end{cases}$$



Graphical interpretation. The unique intersection of the two best-response correspondences is interior: neither player can profitably switch from their equilibrium mixture. There are no boundary intersections, reflecting the absence of pure equilibria.